

# SHARP ADAMS TYPE INEQUALITIES IN SOBOLEV SPACES $W^{m, \frac{n}{m}}(\mathbb{R}^n)$ FOR ARBITRARY INTEGER $m$

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**ABSTRACT.** The main purpose of our paper is to prove sharp Adams-type inequalities in unbounded domains of  $\mathbb{R}^n$  for the Sobolev space  $W^{m, \frac{n}{m}}(\mathbb{R}^n)$  for any positive integer  $m$  less than  $n$ . Our results complement those of Ruf and Sani [28] where such inequalities are only established for even integer  $m$ . Our inequalities are also a generalization of the Adams-type inequalities in the special case  $n = 2m = 4$  proved in [33] and stronger than those in [28] when  $n = 2m$  for all positive integer  $m$  by using different Sobolev norms.

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  be a bounded domain. The Sobolev embedding theorems say that  $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ ,  $1 \leq q \leq \frac{np}{n-kp}$ ,  $kp < n$  and that  $W_0^{k, \frac{n}{k}}(\Omega) \subset L^q(\Omega)$ ,  $1 \leq q < \infty$ . However, we can show by easy examples that  $W_0^{k, \frac{n}{k}}(\Omega) \not\subset L^\infty(\Omega)$ . In this case, Yudovich [32], Pohozaev [26] and Trudinger [31] independently showed that  $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$  where  $L_{\varphi_n}(\Omega)$  is the Orlicz space associated with the Young function  $\varphi_n(t) = \exp(|t|^{n/(n-1)}) - 1$ . In his 1971 paper [25], J. Moser finds the largest positive real number  $\beta_n = n\omega_{n-1}^{\frac{1}{n-1}}$ , where  $\omega_{n-1}$  is the area of the surface of the unit  $n$ -ball, such that if  $\Omega$  is a domain with finite  $n$ -measure in Euclidean  $n$ -space  $\mathbb{R}^n$ ,  $n \geq 2$ , then there is a constant  $c_0$  depending only on  $n$  such that

$$\frac{1}{|\Omega|} \int_{\Omega} \exp\left(\beta |u|^{\frac{n}{n-1}}\right) dx \leq c_0$$

for any  $\beta \leq \beta_n$ , any  $u \in W_0^{1,n}(\Omega)$  with  $\int_{\Omega} |\nabla u|^n dx \leq 1$ . Moreover, this constant  $\beta_n$  is sharp in the meaning that if  $\beta > \beta_n$ , then the above inequality can no longer hold with some  $c_0$  independent of  $u$ . Such an inequality is nowadays known as Moser-Trudinger type inequality.

Moser's result for first order derivatives was extended to high order derivatives by D. Adams [2]. Indeed, Adams found the sharp constants for higher order Moser's type inequality. To state Adams' result, we use the symbol  $\nabla^m u$ ,  $m$  is a positive integer, to denote the  $m$ -th order gradient for  $u \in C^m$ , the class of  $m$ -th order differentiable

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functions:

$$\nabla^m u = \begin{cases} \Delta^{\frac{m}{2}} u & \text{for } m \text{ even} \\ \nabla \Delta^{\frac{m-1}{2}} u & \text{for } m \text{ odd} \end{cases}.$$

where  $\nabla$  is the usual gradient operator and  $\Delta$  is the Laplacian. We use  $\|\nabla^m u\|_p$  to denote the  $L^p$  norm ( $1 \leq p \leq \infty$ ) of the function  $|\nabla^m u|$ , the usual Euclidean length of the vector  $\nabla^m u$ . We also use  $W_0^{k,p}(\Omega)$  to denote the Sobolev space which is a completion of  $C_0^\infty(\Omega)$

under the norm of  $\|u\|_{L^p(\Omega)} + \sum_{j=1}^k \|\nabla^j u\|_{L^p(\Omega)}$ . Then Adams proved the following:

**Theorem A.** *Let  $\Omega$  be an open and bounded set in  $\mathbb{R}^n$ . If  $m$  is a positive integer less than  $n$ , then there exists a constant  $C_0 = C(n, m) > 0$  such that for any  $u \in W_0^{m, \frac{n}{m}}(\Omega)$  and  $\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \leq 1$ , then*

$$\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \leq C_0$$

for all  $\beta \leq \beta(n, m)$  where

$$\beta(n, m) = \begin{cases} \frac{n}{w_{n-1}} \left[ \frac{\pi^{n/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd} \\ \frac{n}{w_{n-1}} \left[ \frac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is even} \end{cases}.$$

Furthermore, for any  $\beta > \beta(n, m)$ , the integral can be made as large as possible.

Note that  $\beta(n, 1)$  coincides with Moser's value of  $\beta_n$  and  $\beta(2m, m) = 2^{2m} \pi^m \Gamma(m+1)$  for both odd and even  $m$ .

The Adams inequality was extended recently by Tarsi [29]. More precisely, Tarsi used the Sobolev space with Navier boundary conditions  $W_N^{m, \frac{n}{m}}(\Omega)$  which contains the Sobolev space  $W_0^{m, \frac{n}{m}}(\Omega)$  as a closed subspace:

**Theorem B.** *Let  $n > 2$  and  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Then there exists a constant  $C_0 = C(n, m) > 0$  such that for any  $u \in W_N^{m, \frac{n}{m}}(\Omega)$  with  $\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \leq 1$*

$$\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \leq C_0$$

for all  $\beta \leq \beta(n, m)$ . Furthermore, the constant  $\beta(n, m)$  is sharp in the sense that if  $\beta > \beta(n, m)$  then the supremum is infinite.

The Adams inequality was also extended to compact Riemannian manifolds without boundary by Fontana [17]. Also, the singular Moser-Trudinger inequalities and the singular Adams inequalities which are the combinations of the Hardy inequalities, Moser-Trudinger inequalities and Adams inequalities are established in [4, 22].

The Moser-Trudinger's inequality and Adams inequality play an essential role in geometric analysis and in the study of the exponential growth partial differential equations where, roughly speaking, the nonlinearity behaves like  $e^{\alpha |u|^{\frac{n}{n-m}}}$  as  $|u| \rightarrow \infty$ . Here we mention Atkinson-Peletier [9], Carleson-Chang [12], Adimurthi et al. [3, 4, 5, 6, 7, 8],

de Figueiredo-Miyagaki-Ruf [14], J.M. do Ó [15], de Figueiredo- do Ó-Ruf [13], Lam-Lu [20, 21] and the references therein.

We notice that when  $\Omega$  has infinite volume, the Moser-Trudinger's inequality and Adams inequality don't make sense since the left hand side is trivial. The sharp Moser-Trudinger type inequality for the first order derivatives in the case  $|\Omega| = +\infty$  was obtained by B. Ruf [27] in dimension two and Y.X. Li-Ruf [23] in general dimension. In fact, such an inequality at the subcritical case was derived earlier by Cao [11] in dimension two and by Adachi and Tanaka in high dimensions [1]. Recently, Ruf and Sani proved the Adams type inequality for **higher derivatives of even orders** when  $\Omega$  has infinite volume. Indeed, Ruf and Sani proved the following Adams type inequality (see [28]):

**Theorem C.** *Let  $m$  be **an even integer** less than  $n$ . There exists a constant  $C_{m,n} > 0$  such that for any domain  $\Omega \subseteq \mathbb{R}^n$*

$$\sup_{u \in W_0^{m, \frac{n}{m}}(\Omega), \|u\|_{m,n} \leq 1} \int_{\Omega} \phi \left( \beta_0(n, m) |u|^{\frac{n}{n-m}} \right) dx \leq C_{m,n}$$

where

$$\begin{aligned} \beta_0(n, m) &= \frac{n}{\omega_{n-1}} \left[ \frac{\pi^{\frac{n}{2}} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)} \right]^{\frac{n}{n-m}}, \\ \phi(t) &= e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2} \frac{t^j}{j!} \\ j_{\frac{n}{m}} &= \min \left\{ j \in \mathbb{N} : j \geq \frac{n}{m} \right\} \geq \frac{n}{m}. \\ \|u\|_{m,n} &= \left\| (-\Delta + I)^{\frac{m}{2}} u \right\|_{\frac{n}{m}} \end{aligned}$$

*This inequality is sharp in the sense that if we replace  $\beta_0(n, m)$  by any  $\beta > \beta_0(n, m)$ , then the supremum is infinite.*

We note that the norm  $\|u\|_{n,m}$  used in Theorem C is equivalent to the Sobolev norm

$$\|u\|_{W^{m, \frac{n}{m}}} = \left( \|u\|_{\frac{n}{m}}^{\frac{n}{m}} + \sum_{j=1}^m \|\nabla^j u\|_{\frac{n}{m}}^{\frac{n}{m}} \right)^{\frac{m}{n}}.$$

In particular, if  $u \in W_0^{m, \frac{n}{m}}(\Omega)$  or  $u \in W^{m, \frac{n}{m}}(\mathbb{R}^n)$ , then  $\|u\|_{W^{m, \frac{n}{m}}} \leq \|u\|_{m,n}$ .

The work of Ruf and Sani raised a good open question: **Does Theorem C hold when  $m$  is odd?**

One of the primary purposes of this paper is to answer the above question in an affirmative way. This is stated as follows:

**Theorem 1.1.** *Let  $m$  be an odd integer less than  $n$ :  $m = 2k + 1$ ,  $k \in \mathbb{N}$  and let  $\beta(n, m)$  be as in Theorem A and the function  $\phi$  be as in Theorem C. Then there holds*

$$\sup_{u \in W^{m, \frac{n}{m}}(\mathbb{R}^n), \left\| \nabla (-\Delta + I)^k u \right\|_{\frac{n}{m}} + \left\| (-\Delta + I)^k u \right\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \phi \left( \beta(n, m) |u|^{\frac{n}{n-m}} \right) dx < \infty.$$

Moreover, the constant  $\beta(n, m)$  is sharp in the sense that if we replace  $\beta(n, m)$  by any  $\beta > \beta(n, m)$ , then the supremum is infinity.

In the special case  $n = 2m$  and  $m$  an arbitrary positive integer, we can prove the following stronger result which is the second main theorem of this paper:

**Theorem 1.2.** *If  $m = 2k + 1$ ,  $k \in \mathbb{N}$ , then for all  $\tau > 0$ , there holds*

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \|\nabla(-\Delta + \tau I)^k u\|_2^2 + \tau \|(-\Delta + \tau I)^k u\|_2^2 \leq 1} \int_{\mathbb{R}^{2m}} \left( e^{\beta(2m, m) u^2} - 1 \right) dx < \infty.$$

*If  $m = 2k$ ,  $k \in \mathbb{N}$ , then for all  $\tau > 0$ , there holds*

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \|(-\Delta + \tau I)^k u\|_2 \leq 1} \int_{\mathbb{R}^{2m}} \left( e^{\beta(2m, m) u^2} - 1 \right) dx < \infty.$$

Moreover, the constant  $\beta(2m, m)$  is sharp in the above inequalities in the sense that if we replace  $\beta(2m, m)$  by any  $\beta > \beta(2m, m)$ , then the supremums will be infinity.

We note that for  $m = 2k + 1$  and any  $a_0 = 1, a_2 > 0, \dots, a_m > 0$ , there is some  $\tau > 0$  such that (see Lemma 2.2):

$$\left\| \nabla (-\Delta + \tau I)^k u \right\|_2^2 + \tau \left\| (-\Delta + \tau I)^k u \right\|_2^2 \leq \sum_{j=0}^m a_{m-j} \int_{\mathbb{R}^n} |\nabla^j u|^2 dx$$

and for  $m = 2k$  and any  $a_0 = 1, a_2 > 0, \dots, a_m > 0$ , there is some  $\tau > 0$  such that (see Lemma 2.1):

$$\left\| (-\Delta + \tau I)^k u \right\|_2^2 \leq \sum_{j=0}^m a_{m-j} \int_{\mathbb{R}^n} |\nabla^j u|^2 dx.$$

Thus, as a consequence, we will be able to establish the third main theorem of this paper. Namely, we will replace the norm  $\|\cdot\|_{m,n}$  by  $\|\cdot\|_{W^{m, \frac{n}{m}}}$  in the above Theorem C in the case  $n = 2m$  for all positive integer  $m$ .

**Theorem 1.3.** *Let  $m \geq 1$  be an integer number. For all constants  $a_0 = 1, a_1, \dots, a_m > 0$ , there holds*

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \int_{\mathbb{R}^{2m}} \left( \sum_{j=0}^m a_{m-j} |\nabla^j u|^2 \right) dx \leq 1} \int_{\mathbb{R}^{2m}} \left[ \exp(\beta(2m, m) |u|^2) - 1 \right] dx < \infty.$$

Furthermore this inequality is sharp, i.e., if  $\beta(2m, m)$  is replaced by any  $\beta > \beta(2m, m)$ , then the supremum is infinite.

In the special case  $n = 2m = 4k = 4$ , the above theorem was proved by Yang in [33].

As a corollary of the above theorem, we have the following Adams type inequality with the standard Sobolev norm:

**Theorem 1.4.** *Let  $m \geq 1$  be an integer number. There holds*

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \|u\|_{W^{m,2}} \leq 1} \int_{\mathbb{R}^{2m}} \left[ \exp(\beta(2m, m) |u|^2) - 1 \right] dx < \infty.$$

Furthermore this inequality is sharp, i.e., if  $\beta(2m, m)$  is replaced by any  $\beta > \beta(2m, m)$ , then the supremum is infinite.

Since the fact that if  $u \in W_0^{m, \frac{n}{m}}(\Omega)$  or  $u \in W^{m, \frac{n}{m}}(\mathbb{R}^n)$ , then  $\|u\|_{W^{m, \frac{n}{m}}} \leq \|u\|_{m, n}$ , our result is stronger than the one in [28] in the case  $m$  is even. Moreover, our theorems still hold when  $m$  is odd.

We organize this paper as follows: In Section 2, we provide some preliminaries. We build an iterated comparison in Section 3 and use it to prove the Adams type inequalities (Theorem 1.2, Theorem 1.3 and Theorem 1.4) for the case  $n = 2m = 4k$ ,  $k \in \mathbb{N}$ , namely when  $m$  is **even** in Section 4. Section 5 is devoted to proving Theorems 1.2, 1.3 and 1.4 when  $n = 2m = 4k + 2$ , namely when  $m$  is **odd**. In fact, we will first prove these theorems in the special case when  $n = 2m = 6$ . Then we will prove these theorems in the general case  $n = 2m = 2(2k + 1)$ . Finally, the Adams-type inequality when  $m$  is odd in general (Theorem 1.1) is proved in Section 6.

## 2. PRELIMINARIES

In this section, we provide some preliminaries. For  $u \in W^{m, 2}(\mathbb{R}^{2m})$  with  $1 \leq p < \infty$ , we will denote by  $\nabla^j u$ ,  $j \in \{1, 2, \dots, m\}$ , the  $j$ -th order gradient of  $u$ , namely

$$\nabla^j u = \begin{cases} \Delta^{\frac{j}{2}} u & \text{for } j \text{ even} \\ \nabla \Delta^{\frac{j-1}{2}} u & \text{for } j \text{ odd} \end{cases}.$$

For  $m = 2k$ ,  $k \in \mathbb{N}$ ,  $\tau > 0$ , we have the following observations:

$$(2.1) \quad (-\Delta + \tau I)^k u = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \tau^i \Delta^{k-i} u$$

where

$$\binom{k}{j} = \frac{k!}{j!(k-j)!}.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^{2m}} |(-\Delta + \tau I)^k u|^2 dx &= \int_{\mathbb{R}^{2m}} \left| \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \tau^i \Delta^{k-i} u \right|^2 dx \\ &= \int_{\mathbb{R}^{2m}} \sum_{0 \leq i, j \leq k} (-1)^{k-i} (-1)^{k-j} \binom{k}{i} \binom{k}{j} \tau^i \tau^j \Delta^{k-i} u \Delta^{k-j} u dx \\ &= \sum_{s=0}^{2k} \sum_{i+j=s} (-1)^{k-i} (-1)^{k-j} \binom{k}{i} \binom{k}{j} \tau^i \tau^j \int_{\mathbb{R}^{2m}} \Delta^{k-i} u \Delta^{k-j} u dx \\ &= \sum_{s=0}^{2k} \sum_{i+j=s} \binom{k}{i} \binom{k}{j} \tau^s \int_{\mathbb{R}^{2m}} |\nabla^{2k-s} u|^2 dx. \end{aligned}$$

From the coefficients of  $x^s$  in the identity

$$(1+x)^k (1+x)^k = (1+x)^{2k}$$

we have

$$\sum_{i+j=s} \binom{k}{i} \binom{k}{j} = \binom{2k}{s}$$

and then

$$(2.2) \quad \int_{\mathbb{R}^{2m}} \left| (-\Delta + \tau I)^k u \right|^2 dx = \int_{\mathbb{R}^{2m}} \left( \sum_{j=0}^m \binom{m}{j} \tau^{m-j} |\nabla^j u|^2 \right) dx.$$

From these observations, we have when  $m = 2k$ ,  $k \in \mathbb{N}$ :

$$(2.3) \quad \left\| (-\Delta + \tau I)^k u \right\|_2 = \left[ \sum_{j=0}^m \binom{m}{j} \tau^{m-j} \left\| \nabla^j u \right\|_2^2 \right]^{1/2}.$$

From (2.1), (2.2) and (2.3), we have

**Lemma 2.1.** *Assume  $m = 2k$ ,  $k \in \mathbb{N}$ . Let  $a_0 = 1, a_1, \dots, a_m > 0$ . There exists a real number  $\tau > 0$  such that for all  $u \in W^{m,2}(\mathbb{R}^{2m})$ :*

$$\left\| (-\Delta + \tau I)^k u \right\|_2^2 \leq \sum_{j=0}^m a_{m-j} \left\| \nabla^j u \right\|_2^2$$

*Proof.* We just need to choose  $\tau > 0$  such that

$$\binom{m}{j} \tau^{m-j} \leq a_{m-j}, \quad j = 0, 1, \dots, m.$$

□

When  $m = 2k + 1$ ,  $k \in \mathbb{N}$ , we have

$$(2.4) \quad \nabla (-\Delta + \tau I)^k u = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \tau^i \nabla \Delta^{k-i} u$$

where

$$\binom{k}{j} = \frac{k!}{j!(k-j)!}.$$

Similarly, we can prove that

$$(2.5) \quad \int_{\mathbb{R}^{2m}} \left| \nabla (-\Delta + \tau I)^k u \right|^2 dx = \int_{\mathbb{R}^{2m}} \left( \sum_{j=1}^m \binom{m-1}{j-1} \tau^{m-j} |\nabla^j u|^2 \right) dx.$$

Thus, we have for  $m = 2k + 1$ ,  $k \in \mathbb{N}$ :

$$(2.6) \quad \left\| \nabla (-\Delta + \tau I)^k u \right\|_2 = \left[ \sum_{j=1}^m \binom{m-1}{j-1} \tau^{m-j} \left\| \nabla^j u \right\|_2^2 \right]^{1/2}.$$

By (2.3) and (2.6), we get

$$(2.7) \quad \left\| \nabla (-\Delta + \tau I)^k u \right\|_2^2 + \tau \left\| (-\Delta + \tau I)^k u \right\|_2^2 = \sum_{j=0}^m \binom{m}{j} \tau^{m-j} \left\| \nabla^j u \right\|_2^2.$$

**Lemma 2.2.** *Assume  $m = 2k + 1$ ,  $k \in \mathbb{N}$ . Let  $a_0 = 1, a_1, \dots, a_m > 0$ . There exists a real number  $\tau > 0$  such that for all  $u \in W^{m,2}(\mathbb{R}^{2m})$ :*

$$\left\| \nabla (-\Delta + \tau I)^k u \right\|_2^2 + \tau \left\| (-\Delta + \tau I)^k u \right\|_2^2 \leq \sum_{j=0}^m a_{m-j} \left\| \nabla^j u \right\|_2^2$$

*Proof.* Again, we just need to choose  $\tau > 0$  such that

$$\binom{m}{j} \tau^{m-j} \leq a_{m-j}.$$

□

In the general case, we have the following result

**Lemma 2.3.** *Assume that  $m$  is an odd integer less than  $n$ :  $m = 2k + 1$ . There exists a real number  $C > 0$  such that for all  $u \in W^{m, \frac{n}{m}}(\mathbb{R}^n)$ :*

$$\left\| \nabla^m u \right\|_{\frac{n}{m}}^{\frac{n}{m}} + \frac{1}{C} \sum_{j=0}^{m-1} \left\| \nabla^j u \right\|_{\frac{n}{m}}^{\frac{n}{m}} \leq \left\| \nabla (-\Delta + I)^k u \right\|_{\frac{n}{m}}^{\frac{n}{m}} + \left\| (-\Delta + I)^k u \right\|_{\frac{n}{m}}^{\frac{n}{m}}.$$

We now introduce the Sobolev space of functions with homogeneous Navier boundary conditions:

$$W_N^{m,2}(B_R) := \left\{ u \in W^{m,2} : \Delta^j u = 0 \text{ on } \partial B_R \text{ for } 0 \leq j \leq \left\lfloor \frac{m-1}{2} \right\rfloor \right\}$$

where  $B_R = \{x \in \mathbb{R}^{2m} : |x| < R\}$ . It is easy to see that  $W_N^{m,2}(B_R)$  contains  $W_0^{m,2}(B_R)$  as a closed subspace. Also, we define

$$\begin{aligned} W_{rad}^{m,2}(B_R) &:= \{u \in W^{m,2} : u(x) = u(|x|) \text{ a.e. in } B_R\}, \\ W_{N,rad}^{m,2}(B_R) &= W_N^{m,2}(B_R) \cap W_{rad}^{m,2}(B_R). \end{aligned}$$

Finally, we give some radial lemmas which will be used in our proofs (see [10, 18, 28]):

**Lemma 2.4.** *If  $u \in W^{1, \frac{n}{m}}(\mathbb{R}^n)$  then*

$$|u(x)| \leq \left( \frac{1}{m\sigma_n} \right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{n}m}} \|u\|_{W^{1, \frac{n}{m}}}$$

for a.e.  $x \in \mathbb{R}^n$ , where  $\sigma_n$  is the volume of the unit ball in  $\mathbb{R}^n$ .

**Lemma 2.5.** *If  $u \in L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , is a radial non-increasing function, then*

$$|u(x)| \leq \left( \frac{n}{\omega_{n-1}} \right)^{\frac{1}{p}} \frac{1}{|x|^{\frac{n}{p}}} \|u\|_{L^p(\mathbb{R}^n)}$$

for a.e.  $x \in \mathbb{R}^n$ .

## 3. AN ITERATED COMPARISON PRINCIPLE

In this section, we still denote by  $B_R$  the set  $\{x \in \mathbb{R}^n : |x| < R\}$  and  $|B_R|$  the Lebesgue measure of  $B_R$ , namely  $|B_R| = \sigma_n R^n$  where  $\sigma_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . Let  $u : B_R \rightarrow \mathbb{R}$  be a measurable function. The distribution function of  $u$  is defined by

$$\mu_u(t) = |\{x \in B_R \mid |u(x)| > t\}|, \quad \forall t \geq 0.$$

The decreasing rearrangement of  $u$  is defined by

$$u^*(s) = \inf \{t \geq 0 : \mu_u(t) < s\}, \quad \forall s \in [0, |B_R|],$$

and the spherically symmetric decreasing rearrangement of  $u$  by

$$u^\#(x) = u^*(\sigma_n |x|^n) \quad \forall x \in B_R.$$

We have that  $u^\#$  is the unique nonnegative integrable function which is radially symmetric, nonincreasing and has the same distribution function as  $|u|$ .

Let  $\tau > 0$  and  $u$  be a weak solution of

$$(3.1) \quad \begin{cases} -\Delta u + \tau u = f & \text{in } B_R \\ u \in W_0^{1,2}(B_R) \end{cases}$$

where  $f \in L^{\frac{2n}{n+2}}(B_R)$ . We have the following result that can be found in [30]:

**Proposition 3.1.** *If  $u$  is a nonnegative weak solution of (3.1) then*

$$(3.2) \quad -\frac{du^*}{ds}(s) \leq \frac{s^{\frac{2}{n}-2}}{n^2 \sigma_n^{2/n}} \int_0^s (f^* - \tau u^*) d\tau, \quad \forall s \in (0, |B_R|).$$

Now, we consider the problem

$$(3.3) \quad \begin{cases} -\Delta v + \tau v = f^\# & \text{in } B_R \\ v \in W_0^{1,2}(B_R) \end{cases}$$

Due to the radial symmetry of the equation, the unique solution  $v$  of (3.3) is radially symmetric and we have

$$-\frac{d\widehat{v}}{ds}(s) = \frac{s^{\frac{2}{n}-2}}{n^2 \sigma_n^{2/n}} \int_0^s (f^* - \tau \widehat{v}) d\tau, \quad \forall s \in (0, |B_R|)$$

where  $\widehat{v}(\sigma_n |x|^n) := v(x)$ . We have the following comparison of integrals in balls that again can be found in [30]:

**Proposition 3.2.** *Let  $u, v$  be weak solutions of (3.1) and (3.3) respectively. For every  $r \in (0, R)$  we have*

$$\int_{B_r} u^\# dx \leq \int_{B_r} v dx.$$

We now apply the comparison principle for the polyharmonic operator. Let  $u \in W^{2k,2}(B_R)$  be a weak solution of

$$(3.4) \quad \begin{cases} (-\Delta + \tau I)^k u = f & \text{in } B_R \\ u \in W_N^{2k,2}(B_R) \end{cases}$$



where  $f \in L^{\frac{2n}{n+2}}(B_R)$ . If we consider the problem

$$(3.5) \quad \begin{cases} (-\Delta + \tau I)^k v = f^\# \text{ in } B_R \\ v \in W_N^{2k,2}(B_R) \end{cases}$$

then we have the following comparison of integrals in balls:

**Proposition 3.3.** *Let  $u, v$  be weak solutions of the polyharmonic problems (3.4) and (3.5) respectively. For every  $r \in (0, R)$  we have*

$$\int_{B_r} u^\# dx \leq \int_{B_r} v dx.$$

*Proof.* Since equations in (3.4) and (3.5) are considered with homogeneous Navier boundary conditions, they may be rewritten as second order systems:

$$\begin{aligned} (P1) \begin{cases} -\Delta u_1 + \tau u_1 = f \text{ in } B_R \\ u_1 \in W_0^{1,2}(B_R) \end{cases} & \quad (Pi) \begin{cases} -\Delta u_i + \tau u_i = u_{i-1} \text{ in } B_R \\ u_i \in W_0^{1,2}(B_R) \end{cases} \quad i \in \{2, 3, \dots, k\} \\ (Q1) \begin{cases} -\Delta v_1 + \tau v_1 = f^\# \text{ in } B_R \\ v_1 \in W_0^{1,2}(B_R) \end{cases} & \quad (Qi) \begin{cases} -\Delta v_i + \tau v_i = v_{i-1} \text{ in } B_R \\ v_i \in W_0^{1,2}(B_R) \end{cases} \quad i \in \{2, 3, \dots, k\} \end{aligned}$$

where  $u_k = u$  and  $v_k = v$ . Thus we have to prove that for every  $r \in (0, R)$

$$(3.6) \quad \int_{B_r} u_k^\# dx \leq \int_{B_r} v_k dx.$$

By the above proposition (Proposition 3.2), we have

$$\int_{B_r} u_1^\# dx \leq \int_{B_r} v_1 dx.$$

Now, if we have

$$\int_{B_r} u_j^\# dx \leq \int_{B_r} v_j dx \text{ for all } j = 1, \dots, i,$$

we will prove that

$$\int_{B_r} u_{i+1}^\# dx \leq \int_{B_r} v_{i+1} dx.$$

Without loss of generality, we may assume that  $u_{i+1} \geq 0$ . In fact, let  $\bar{u}_{i+1}$  be a weak solution of

$$\begin{cases} -\Delta \bar{u}_{i+1} + \tau \bar{u}_{i+1} = |u_i| \text{ in } B_R \\ \bar{u}_{i+1} \in W_0^{1,2}(B_R) \end{cases}$$

then the maximum principle implies that  $\bar{u}_{i+1} \geq 0$  and  $\bar{u}_{i+1} \geq |u_{i+1}|$  in  $B_R$ .

Since  $u_{i+1}$  is a nonnegative weak solution of  $(P(i+1))$  and  $v_{i+1}$  is a nonnegative weak solution of  $(Q(i+1))$  then by Proposition 3.1 we have

$$\begin{aligned} -\frac{du_{i+1}^*}{ds}(s) &\leq \frac{s^{\frac{2}{n}-2}}{n^2 \sigma_n^{2/n}} \int_0^s (u_i^* - \tau u_{i+1}^*) d\tau, \quad \forall s \in (0, |B_R|), \\ -\frac{d\widehat{v}_{i+1}}{ds}(s) &= \frac{s^{\frac{2}{n}-2}}{n^2 \sigma_n^{2/n}} \int_0^s (\widehat{v}_i - \tau \widehat{v}_{i+1}) d\tau, \quad \forall s \in (0, |B_R|) \end{aligned}$$

Thus for all  $s \in (0, |B_R|)$

$$\frac{d\widehat{v}_{i+1}}{ds}(s) - \frac{du_{i+1}^*}{ds}(s) - \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s (\tau \widehat{v}_{i+1} - \tau u_{i+1}^*) d\tau \leq \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s (u_i^* - \widehat{v}_i) d\tau.$$

Using the induction hypotheses, we get that

$$\int_0^s (u_i^* - \widehat{v}_i) d\tau \leq 0 \quad \forall s \in (0, |B_R|)$$

and then

$$\frac{d\widehat{v}_{i+1}}{ds}(s) - \frac{du_{i+1}^*}{ds}(s) - \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s (\tau \widehat{v}_{i+1} - \tau u_{i+1}^*) d\tau \leq 0.$$

Setting

$$y(s) = \int_0^s (\widehat{v}_{i+1} - u_{i+1}^*) d\tau \quad \forall s \in (0, |B_R|)$$

we get

$$\begin{cases} y'' - \frac{\tau s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} y \leq 0, \quad \forall s \in (0, |B_R|) \\ y(0) = y'(|B_R|) = 0 \end{cases}.$$

By maximum principle, we have that  $y \geq 0$  which is the desired result.  $\square$

From the above proposition, we have the following corollary:

**Corollary 3.1.** *Let  $u, v$  be weak solutions of the polyharmonic problems (3.4) and (3.5) respectively. For every convex nondecreasing function  $\phi : [0, +\infty) \rightarrow [0, +\infty)$  we have*

$$\int_{B_r} \phi(|u|) dx \leq \int_{B_r} \phi(|v|) dx.$$

**Remark 3.1.** *If  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $\text{supp} f \subset B_R$ , then we can conclude that  $u$  and  $v$  in Proposition 3.3 belong to  $W_N^{m, \frac{n}{m}}(B_R)$  with  $m = 2k$  or  $2k + 1$ .*

#### 4. PROOFS OF THEOREMS 1.2, 1.3 AND 1.4 WHEN $m$ IS EVEN

In this section, we will prove Theorem 1.2 in the case when  $m$  is even, namely,  $m = 2k$ ,  $k \in \mathbb{N}$ .

**Theorem 4.1.** *Let  $m = 2k$ ,  $k \in \mathbb{N}$ . For all  $\tau > 0$ , there holds*

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \|(-\Delta + \tau I)^k u\|_2 \leq 1} \int_{\mathbb{R}^{2m}} \left( e^{\beta(2m,m)u^2} - 1 \right) dx < \infty.$$

*Furthermore this inequality is sharp, i.e., if  $\beta(2m,m)$  is replaced by any  $\beta > \beta(2m,m)$ , then the supremum is infinite.*

*Proof.* Let  $u \in W^{m,2}(\mathbb{R}^{2m})$ ,  $\|(-\Delta + \tau I)^k u\|_2 \leq 1$ , by the fact that  $C_0^\infty(\mathbb{R}^{2m})$  is dense in  $W^{m,2}(\mathbb{R}^{2m})$ , without loss of generality, we can find a sequence of functions  $u_l \in C_0^\infty(\mathbb{R}^{2m})$  such that  $u_l \rightarrow u$  in  $W^{m,2}(\mathbb{R}^{2m})$  and  $\int_{\mathbb{R}^{2m}} |(-\Delta + \tau I)^k u_l|^2 dx \leq 1$  and suppose that  $\text{supp } u_l \subset B_{R_l}$  for any fixed  $l$ . Let  $f_l := (-\Delta + \tau I)^k u_l$ . Consider the problem

$$\begin{cases} (-\Delta + \tau I)^k v_l = f_l^\# \\ v_l \in W_N^{m,2}(B_{R_l}) \end{cases}.$$

By the property of rearrangement, we have

$$(4.1) \quad \int_{B_{R_l}} |(-\Delta + \tau I)^k v_l|^2 dx = \int_{B_{R_l}} |(-\Delta + \tau I)^k u_l|^2 dx \leq 1$$

and by Corollary 3.1, we get

$$\int_{B_{R_l}} (e^{\beta_0 u_l^2} - 1) dx = \int_{B_{R_l}} (e^{\beta_0 u_l^{\#2}} - 1) dx \leq \int_{B_{R_l}} (e^{\beta_0 v_l^2} - 1) dx.$$

Also, from (4.1) and (2.3), we have

$$\begin{aligned} \|v_l\|_{W^{1,2}} &= \left[ \int_{B_{R_l}} (|v_l|^2 + |\nabla v_l|^2) \right]^{1/2} \\ &\leq \sqrt{\frac{1}{\tau^m} + \frac{1}{m\tau^{m-1}}}. \end{aligned}$$

Now, writing

$$\begin{aligned} \int_{B_{R_l}} (e^{\beta_0 v_l^2} - 1) dx &\leq \int_{B_{R_0}} (e^{\beta_0 v_l^2} - 1) dx + \int_{B_{R_l} \setminus B_{R_0}} (e^{\beta_0 v_l^2} - 1) dx \\ &= I_1 + I_2 \end{aligned}$$

where  $R_0$  depends only on  $\tau$  and will be chosen later, we will prove that both  $I_1$  and  $I_2$  are bounded uniformly by a constant that depends only on  $\tau$ .

Using Theorem B, we can estimate  $I_1$ . Indeed, we just need to construct an auxiliary radial function  $w_l \in W_N^{m,2}(B_{R_0})$  with  $\|\nabla^m w_l\|_2 \leq 1$  which increases the integral we are interested in. Such a function was constructed in [28]. For the completeness, we give the detail here. For each  $i \in \{1, 2, \dots, k-1\}$  we define

$$g_i(|x|) := |x|^{m-2i}, \quad \forall x \in B_{R_0}$$

so  $g_i \in W_{rad}^{m,2}(B_{R_0})$ . Moreover,

$$\Delta^j g_i(|x|) = \begin{cases} c_i^j |x|^{m-2(i+j)} & \text{for } j \in \{1, 2, \dots, k-i\} \\ 0 & \text{for } j \in \{k-i+1, \dots, k\} \end{cases} \quad \forall x \in B_{R_0}$$

where

$$c_i^j = \prod_{h=1}^j [n + m - 2(h+i)] [m - 2(i+h-1)], \quad \forall j \in \{1, 2, \dots, k-i\}.$$

Let

$$z_l(|x|) := v_l(|x|) - \sum_{i=1}^{k-1} a_i g_i(|x|) - a_k \quad \forall x \in B_{R_0}$$

where

$$a_i := \frac{\Delta^{k-i} v_l(R_0) - \sum_{j=1}^{i-1} a_j \Delta^{k-i} g_j(R_0)}{\Delta^{k-i} g_i(R_0)}, \quad \forall i \in \{1, 2, \dots, k-1\},$$

$$a_k := v_l(R_0) - \sum_{i=1}^{k-1} a_i g_i(R_0).$$

We can check that (see [28])

$$z_l \in W_{N,rad}^{m,2}(B_{R_0}),$$

$$\nabla^m v_l = \nabla^m z_l \text{ in } B_{R_0}.$$

We have the following lemma whose proof can be found in [28]:

**Lemma 4.1.** *For  $0 < |x| \leq R_0$  we have for some  $d(m, R_0)$  only depending on  $m$  and  $R_0$  such that*

$$|v_l(|x|)|^2 \leq |z_l(|x|)|^2 \left( 1 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0^{4j-1}} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0^{2m-1}} \|v_l\|_{W^{1,2}}^2 \right)^2 + d(m, R_0).$$

Now, setting

$$w_l(|x|) := z_l(|x|) \left( 1 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0^{4j-1}} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0^{2m-1}} \|v_l\|_{W^{1,2}}^2 \right).$$

Since

$$z_l \in W_{N,rad}^{m,2}(B_{R_0}),$$

$$\nabla^m v_l = \nabla^m z_l \text{ in } B_{R_0}.$$

we have

$$w_l \in W_{N,rad}^{m,2}(B_{R_0})$$

and

$$\|\nabla^m w_l\|_2 = \|\nabla^m z_l\|_2 \left( 1 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0^{4j-1}} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0^{2m-1}} \|v_l\|_{W^{1,2}}^2 \right).$$

Note that

$$\begin{aligned}\|\nabla^m z_l\|_2 &= \|\nabla^m v_l\|_2 \\ &\leq \left(1 - \lambda \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 - \lambda \|v_l\|_{W^{1,2}}^2\right)^{1/2} \\ &\leq 1 - \frac{\lambda}{2} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 - \frac{\lambda}{2} \|v_l\|_{W^{1,2}}^2\end{aligned}$$

where

$$\lambda = \min \left\{ \binom{m}{j} \tau^{m-j} : j = 0, 1, \dots, m-1 \right\}$$

we have

$$\begin{aligned}\|\nabla^m w_l\|_2 &\leq \left(1 - \frac{\lambda}{2} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 - \frac{\lambda}{2} \|v_l\|_{W^{1,2}}^2\right) \times \\ &\quad \times \left(1 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0^{4j-1}} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0^{2m-1}} \|v_l\|_{W^{1,2}}^2\right) \\ &\leq 1 + \sum_{j=1}^{k-1} \left(\frac{c_m}{R_0^{4j-1}} - \frac{\lambda}{2}\right) \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \left(\frac{c_m}{R_0^{2m-1}} - \frac{\lambda}{2}\right) \|v_l\|_{W^{1,2}}^2 \\ &\leq 1\end{aligned}$$

if we choose  $R_0 = R_0(\tau)$  sufficiently large.

Finally, note that

$$I_1 \leq e^{\beta_0 d(m, R_0)} \int_{B_{R_0}} e^{\beta_0 w_l^2} dx,$$

using Theorem B, we can conclude that  $I_1$  is bounded by a constant depending only on  $\tau$  since  $\|\nabla^m w_l\|_2 \leq 1$  and  $w_l \in W_{N,rad}^{m,2}(B_{R_0})$ .

Now, we will estimate  $I_2$ . We choose  $R_0 \geq \left[\frac{1}{m\sigma_n} \left(\frac{1}{\tau^m} + \frac{1}{m\tau^{m-1}}\right)\right]^{\frac{1}{n-1}}$  then from the Radial Lemma 2.4 we get that  $|v_l(x)| \leq 1$  when  $|x| \geq R_0$ . Thus we have

$$\begin{aligned}I_2 &= \int_{B_{R_l} \setminus B_{R_0}} (e^{\beta_0 v_l^2} - 1) dx \\ &\leq \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!} \int_{B_{R_l}} v_l^2 \\ &\leq \frac{1}{\tau^m} \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!}.\end{aligned}$$

Thus we have that  $\int_{B_{R_l}} (e^{\beta_0 v_l^2} - 1) dx$  is bounded by a constant depending only on  $\tau$ .

Combining the above estimates and using Fatou's lemma, we can conclude that

$$\sup_{\|(-\Delta + \tau I)^k u\|_2 \leq 1} \int_{\mathbb{R}^{2m}} \left( e^{\beta_0 u^2} - 1 \right) dx < \infty.$$

When  $\beta > \beta_0$ , it's easy to check that the sequence given by Ruf and Sani (see Proposition 6.2. in [28]) will make our supremum blow up and we then complete the proof of Theorem 4.1.  $\square$

**Proof of Theorem 1.3 when  $m$  is even:** Choose  $\tau > 0$  as in Lemma 2.1, we have

$$\int_{\mathbb{R}^{2m}} \left( \sum_{j=0}^m a_{m-j} |\nabla^j u|^2 \right) dx \geq \|(-\Delta + \tau I)^k u\|_2^2$$

and then

$$\sup_{\int_{\mathbb{R}^{2m}} \left( \sum_{j=0}^m a_{m-j} |\nabla^j u|^2 \right) dx \leq 1} \int_{\mathbb{R}^{2m}} \left( e^{\beta_0 u^2} - 1 \right) dx \leq \sup_{\|(-\Delta + \tau I)^k u\|_2 \leq 1} \int_{\mathbb{R}^{2m}} \left( e^{\beta_0 u^2} - 1 \right) dx.$$

Furthermore, we can check that the sequence given by Ruf and Sani (see Proposition 6.2. in [28]) will make the supremum in Theorem 1.3 becomes infinite and we complete the proof of Theorem 1.3.

**Proof of Theorem 1.4 when  $m$  is even:** If we choose  $a_i = 1$ ,  $i = 0, 1, \dots, m$ , then by Lemma 2.1 we have proved Theorem 1.4 in the case  $m = 2k$ .

## 5. PROOFS OF THEOREMS 1.2, 1.3 AND 1.4 WHEN $m$ IS ODD

**5.1. Proofs of Theorems 1.2, 1.3 and 1.4 when  $n = 2m = 6$ .** For the convenience, first, we will prove Theorem 1.2 in the special case  $k = 1$ , i.e., we will prove that all  $\tau > 0$ , there holds

$$\sup_{u \in W^{3,2}(\mathbb{R}^6), \|\nabla(-\Delta + \tau I)u\|_2^2 + \tau \|(-\Delta + \tau I)u\|_2^2 \leq 1} \int_{\mathbb{R}^6} \left( e^{\beta_0 u^2} - 1 \right) dx < \infty.$$

where  $\beta_0 = \beta(6, 3)$ .

*Proof.* Let  $u \in W^{3,2}(\mathbb{R}^6)$  be such that

$$\|\nabla(-\Delta + \tau I)u\|_2^2 + \tau \|(-\Delta + \tau I)u\|_2^2 \leq 1.$$

Again, by the density of  $C_0^\infty(\mathbb{R}^6)$  in  $W^{3,2}(\mathbb{R}^6)$ , there exists a sequence of functions  $u_l \in C_0^\infty(\mathbb{R}^6)$ :  $u_l \rightarrow u$  in  $W^{3,2}(\mathbb{R}^6)$ ,

$$\int_{\mathbb{R}^6} |\nabla(-\Delta + \tau I)u_l|^2 + \tau |(-\Delta + \tau I)u_l|^2 dx \leq 1$$

and  $\text{supp } u_l \subset B_{R_l}$  for any fixed  $l$ . Set  $f_l := (-\Delta + \tau I)u_l$  and consider the problem

$$\begin{cases} (-\Delta + \tau I)v_l = f_l^\# \\ v_l \in W_N^{3,2}(B_{R_l}) \end{cases}.$$

By the properties of rearrangement, we have

$$\begin{aligned} \int_{B_{R_l}} |f_l|^2 dx &= \int_{B_{R_l}} |f_l^\#|^2 dx \\ \int_{B_{R_l}} |\nabla f_l^\#|^2 dx &\leq \int_{B_{R_l}} |\nabla f_l|^2 dx \end{aligned}$$

which thus

$$\begin{aligned} \int_{B_{R_l}} |(-\Delta + \tau I) v_l|^2 dx &= \int_{B_{R_l}} |(-\Delta + \tau I) u_l|^2 dx \\ \int_{B_{R_l}} |\nabla (-\Delta + \tau I) v_l|^2 dx &\leq \int_{B_{R_l}} |\nabla (-\Delta + \tau I) u_l|^2 dx \end{aligned}$$

So, we have

$$\begin{aligned} (5.1) \quad & \int_{\mathbb{R}^6} |\nabla (-\Delta + \tau I) v_l|^2 + \tau |(-\Delta + \tau I) v_l|^2 dx \\ & \leq \int_{\mathbb{R}^6} |\nabla (-\Delta + \tau I) u_l|^2 + \tau |(-\Delta + \tau I) u_l|^2 dx \\ & \leq 1. \end{aligned}$$

By the comparison argument (Corollary 3.1), we have

$$\int_{B_{R_l}} \left( e^{\beta_0 u_l^2} - 1 \right) dx = \int_{B_{R_l}} \left( e^{\beta_0 u_l^{\#2}} - 1 \right) dx \leq \int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) dx$$

Recall

$$\begin{aligned} (5.2) \quad & \int_{\mathbb{R}^6} |\nabla (-\Delta + \tau I) v_l|^2 + \tau |(-\Delta + \tau I) v_l|^2 dx \\ & = \|\nabla^3 v_l\|_2^2 + 3\tau \|\Delta v_l\|_2^2 + 3\tau^2 \|\nabla v_l\|_2^2 + \tau^3 \|v_l\|_2^2. \end{aligned}$$

From (5.1) and (5.2), we have

$$\begin{aligned} \|v_l\|_{W^{1,2}} &= \left[ \int_{B_{R_l}} (|v_l|^2 + |\nabla v_l|^2) \right]^{1/2} \\ &\leq \sqrt{\frac{1}{\tau^3} + \frac{1}{3\tau^2}}. \end{aligned}$$

Now, write

$$\begin{aligned} \int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) dx &\leq \int_{B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) dx + \int_{B_{R_l} \setminus B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) dx \\ &= I_1 + I_2 \end{aligned}$$

where  $R_0$  depends only on  $\tau$  and will be chosen later. We will prove that both  $I_1$  and  $I_2$  are bounded uniformly by a constant that depends only on  $\tau$ .

First, we will prove that  $I_1$  is bounded by a constant depending only on  $\tau$  using Theorem B. In order to do that, we will construct an auxiliary radial function  $w_l$  such that  $w_l \in W_N^{3,2}(B_{R_0})$ ,  $\|\nabla^3 w_l\|_2 \leq 1$  and

$$\int_{B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) dx \leq C(R_0) \int_{B_{R_0}} e^{\beta_0 w_l^2} dx.$$

The way to construct this radial function  $w_l$  is very similar to the case when  $m$  is even. Let

$$z_l(|x|) = v_l(|x|) - \frac{\Delta v_l(R_0)}{12} |x|^2 + \frac{R_0^2 \Delta v_l(R_0)}{12} - v_l(R_0), \quad \forall x \in B_{R_0}$$

then  $z_l \in W_{N,rad}^{3,2}(B_{R_0})$ . Similar to that in the proof of Lemma 4.1, and by a combination of Radial Lemmas 2.4 and 2.5, we can prove that for  $0 < |x| \leq R_0$  ( $R_0 > 1$ ), there exists a universal constant  $c > 0$  and a positive constant  $d(R_0)$  depending only on  $R_0$  such that

$$(5.3) \quad |v_l(|x|)|^2 \leq |z_l(|x|)|^2 \left( 1 + c \frac{1}{R_0} \|\Delta v_l\|_2^2 + \frac{c}{R_0} \|v_l\|_{W^{1,2}}^2 \right)^2 + d(R_0).$$

Indeed, we have

$$\begin{aligned} v_l(|x|) &= z_l(|x|) + \frac{\Delta v_l(R_0)}{12} |x|^2 - \frac{R_0^2 \Delta v_l(R_0)}{12} + v_l(R_0) \\ &= z_l(|x|) + g(|x|), \quad \forall x \in B_{R_0} \end{aligned}$$

where

$$g(|x|) = \frac{\Delta v_l(R_0)}{12} |x|^2 - \frac{R_0^2 \Delta v_l(R_0)}{12} + v_l(R_0).$$

Then

$$\begin{aligned} |v_l(|x|)|^2 &= [z_l(|x|) + g(|x|)]^2 \\ &= |z_l(|x|)|^2 + 2z_l(|x|)g(|x|) + |g(|x|)|^2 \\ &\leq |z_l(|x|)|^2 + |z_l(|x|)|^2 |g(|x|)|^2 + 1 + |g(|x|)|^2 \\ &= |z_l(|x|)|^2 (1 + |g(|x|)|^2) + 1 + |g(|x|)|^2. \end{aligned}$$

Note that for  $0 < |x| \leq R_0$  ( $R_0 > 1$ ), we have by Radial lemmas 2.4 and 2.5:

$$\begin{aligned} |g(|x|)| &= \left| \frac{\Delta v_l(R_0)}{12} |x|^2 - \frac{R_0^2 \Delta v_l(R_0)}{12} + v_l(R_0) \right| \\ &\leq \left| \frac{\Delta v_l(R_0)}{12} \right| |x|^2 + \left| \frac{R_0^2 \Delta v_l(R_0)}{12} \right| + |v_l(R_0)| \\ &\leq c R_0^2 \frac{1}{R_0^3} \|\Delta v_l\|_2 + c \frac{1}{R_0^{5/2}} \|v_l\|_{W^{1,2}}^2. \end{aligned}$$

Thus (5.3) follows.

Setting

$$w_l(|x|) := z_l(|x|) \left( 1 + c \frac{1}{R_0} \|\Delta v_l\|_2^2 + \frac{c}{R_0} \|v_l\|_{W^{1,2}}^2 \right), \quad \forall x \in B_{R_0}$$



then it is clear that  $w_l \in W_{N,rad}^{3,2}(B_{R_0})$ . Moreover, we have the following inequalities

$$\begin{aligned}
\|\nabla^3 w_l\|_2 &= \|\nabla^3 z_l\|_2 \left(1 + c \frac{1}{R_0} \|\Delta v_l\|_2^2 + \frac{c}{R_0} \|v_l\|_{W^{1,2}}^2\right) \\
&= \|\nabla^3 v_l\|_2 \left(1 + c \frac{1}{R_0} \|\Delta v_l\|_2^2 + \frac{c}{R_0} \|v_l\|_{W^{1,2}}^2\right) \\
&\leq (1 - 3\tau \|\Delta v_l\|_2^2 - 3\tau^2 \|\nabla v_l\|_2^2 - \tau^3 \|v_l\|_2^2)^{1/2} \left(1 + c \frac{1}{R_0} \|\Delta v_l\|_2^2 + \frac{c}{R_0} \|v_l\|_{W^{1,2}}^2\right) \\
&\leq \left(1 - \frac{3\tau}{2} \|\Delta v_l\|_2^2 - \frac{3\tau^2}{2} \|\nabla v_l\|_2^2 - \frac{\tau^3}{2} \|v_l\|_2^2\right) \left(1 + c \frac{1}{R_0} \|\Delta v_l\|_2^2 + \frac{c}{R_0} \|v_l\|_{W^{1,2}}^2\right) \\
&\leq 1
\end{aligned}$$

if we choose  $R_0$  sufficiently large. Furthermore,

$$\int_{B_{R_0}} (e^{\beta_0 v_l^2} - 1) dx \leq C(R_0) \int_{B_{R_0}} e^{\beta_0 w_l^2} dx.$$

Thus by Theorem B, we have that  $I_1$  is bounded by a constant depending only on  $\tau$ .

Now, we will estimate  $I_2$ . We choose  $R_0 \geq \left[\frac{1}{3\sigma_6} \left(\frac{1}{\tau^3} + \frac{1}{3\tau^2}\right)\right]^{\frac{1}{5}}$  then from the Radial lemma 2.4, we get that  $|v_l(x)| \leq 1$  when  $|x| \geq R_0$ . Thus we have

$$\begin{aligned}
I_2 &= \int_{B_{R_l} \setminus B_{R_0}} (e^{\beta_0 v_l^2} - 1) dx \\
&\leq \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!} \int_{B_{R_l}} v_l^2 \\
&\leq \frac{1}{\tau^3} \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!}.
\end{aligned}$$

Thus we have that  $\int_{B_{R_l}} (e^{\beta_0 v_l^2} - 1) dx$  is bounded by a constant depending only on  $\tau$ .

Combining the above estimates and using Fatou's lemma, we can conclude that

$$\sup_{\|\nabla(-\Delta + \tau I)u\|_2^2 + \tau \|(-\Delta + \tau I)u\|_2^2 \leq 1} \int_{\mathbb{R}^6} (e^{\beta_0 u^2} - 1) dx < \infty.$$

This completes the proof of Theorem 1.2. □

### Proofs of Theorem 1.3 and Theorem 1.4 when $n = 2m = 6$ :

To prove Theorem 1.3 when  $m$  is odd, it suffices to choose  $\tau > 0$  as in Lemma 2.2. Then we have

$$\int_{\mathbb{R}^6} \left( \sum_{j=0}^3 a_{m-j} |\nabla^j u|^2 \right) dx \geq \|\nabla(-\Delta + \tau I)u\|_2^2 + \tau \|(-\Delta + \tau I)u\|_2^2$$

and we get

$$\sup_{\int_{\mathbb{R}^6} \left( \sum_{j=0}^3 a_{m-j} |\nabla^j u|^2 \right) dx \leq 1} \int_{\mathbb{R}^6} \left( e^{\beta_0 u^2} - 1 \right) dx \leq \sup_{\|\nabla(-\Delta + \tau I)u\|_2^2 + \tau \|(-\Delta + \tau I)u\|_2^2 \leq 1} \int_{\mathbb{R}^6} \left( e^{\beta_0 u^2} - 1 \right) dx$$

When  $\beta > \beta_0$ , it is showed by Kozono, Sato and Wadade [19] and Proposition 6.2 in [28] that the supremum in Theorem 1.3 is infinite. In fact, the sequence of test functions which gives the sharpness of Adams' inequality in bounded domains in [2] gives also the sharpness of Adams' inequality in unbounded domains. This completes the proof of Theorem 1.3.

Moreover, we can choose  $a_0 = a_1 = a_2 = a_3 = 1$  to get Theorem 1.4.

**5.2. Proof of Theorem 1.2 when  $m = 2k + 1$ ,  $k \in \mathbb{N}$ .** The idea to prove the Adams type inequality in this case is a combination of ideas in the previous subsection and ideas in Section 4.

*Proof.* Let  $u \in W^{m,2}(\mathbb{R}^{2m})$  be such that

$$\left\| \nabla (-\Delta + \tau I)^k u \right\|_2^2 + \tau \left\| (-\Delta + \tau I)^k u \right\|_2^2 \leq 1.$$

By density arguments, we can find a sequence of functions  $u_l \in C_0^\infty(\mathbb{R}^{2m})$  such that  $u_l \rightarrow u$  in  $W^{m,2}(\mathbb{R}^{2m})$ ,  $\int_{\mathbb{R}^{2m}} \left( \left| \nabla (-\Delta + \tau I)^k u_l \right|^2 + \tau \left| (-\Delta + \tau I)^k u_l \right|^2 \right) dx \leq 1$  and  $\text{supp } u_l \subset B_{R_l}$  for any fixed  $l$ . Let  $f_l := (-\Delta + \tau I)^k u_l$ . Consider the problem

$$\begin{cases} (-\Delta + \tau I)^k v_l = f_l^\# \\ v_l \in W_N^{m,2}(B_{R_l}) \end{cases}.$$

Such a  $v_l$  does exist by Section 3 and Remark 3.1. Moreover, by the properties of rearrangement, we have

$$\begin{aligned} \int_{B_{R_l}} \left| (-\Delta + \tau I)^k v_l \right|^2 dx &= \int_{B_{R_l}} \left| (-\Delta + \tau I)^k u_l \right|^2 dx \\ \int_{B_{R_l}} \left| \nabla (-\Delta + \tau I)^k v_l \right|^2 dx &\leq \int_{B_{R_l}} \left| \nabla (-\Delta + \tau I)^k u_l \right|^2 dx \end{aligned}$$

which leads to

$$(5.4) \quad \int_{\mathbb{R}^{2m}} \left( \left| \nabla (-\Delta + \tau I)^k v_l \right|^2 + \tau \left| (-\Delta + \tau I)^k v_l \right|^2 \right) dx \leq 1$$

Note that from (5.4) and the formula (2.7), we have

$$\begin{aligned} \|v_l\|_{W^{1,2}} &= \left[ \int_{B_{R_l}} (|v_l|^2 + |\nabla v_l|^2) \right]^{1/2} \\ &\leq \sqrt{\frac{1}{\tau^m} + \frac{1}{m\tau^{m-1}}}. \end{aligned}$$

By Corollary 3.1, we get

$$\int_{B_{R_l}} \left( e^{\beta_0 u_l^2} - 1 \right) dx = \int_{B_{R_l}} \left( e^{\beta_0 u_l^{\#2}} - 1 \right) dx \leq \int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) dx.$$

Here,  $\beta_0 = \beta(2m, m)$ .

Again, we write

$$\begin{aligned} \int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) dx &\leq \int_{B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) dx + \int_{B_{R_l} \setminus B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) dx \\ &= I_1 + I_2 \end{aligned}$$

where  $R_0$  depends only on  $\tau$  and will be chosen later. We will prove that both  $I_1$  and  $I_2$  are bounded uniformly by a constant that depends only on  $\tau$ .

First, we will estimate  $I_2$ . We choose  $R_0 \geq \left[ \frac{1}{m\sigma_n} \left( \frac{1}{\tau^m} + \frac{1}{m\tau^{m-1}} \right) \right]^{\frac{1}{n-1}}$  then from the Radial lemma 2.4, we get that  $|v_l(x)| \leq 1$  when  $|x| \geq R_0$ . Thus we have

$$\begin{aligned} I_2 &= \int_{B_{R_l} \setminus B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) dx \\ &\leq \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!} \int_{B_{R_l}} v_l^2 \\ &\leq \frac{1}{\tau^m} \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!}. \end{aligned}$$

Thus we have that  $\int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) dx$  is bounded by a constant depending only on  $\tau$ .

To estimate  $I_1$ , again, we need to construct an auxiliary radial function  $w_l \in W_N^{m,2}(B_{R_0})$  with  $\|\nabla^m w_l\|_2 \leq 1$  which increases the integral we are interested in. We will construct such the function by the very similar way as in the case  $m$  is even [28] and the case  $m = 3$ . For each  $i \in \{0, 1, 2, \dots, k-1\}$  we define

$$g_i(|x|) := |x|^{m-1-2i}, \quad \forall x \in B_{R_0}$$

so  $g_i \in W_{rad}^{m,2}(B_{R_0})$ . Moreover,

$$\Delta^j g_i(|x|) = \begin{cases} c_i^j |x|^{m-1-2(i+j)} & \text{for } j \in \{1, 2, \dots, k-i\} \\ 0 & \text{for } j \in \{k-i+1, \dots, k\} \end{cases} \quad \forall x \in B_{R_0}$$

where

$$c_i^j = \prod_{h=1}^j [6k-2(i+h-1)][2k-2(i+h-1)], \quad \forall j \in \{1, 2, \dots, k-i\}.$$

Let

$$z_l(|x|) := v_l(|x|) - \sum_{i=0}^{k-1} a_i g_i(|x|) - a_k, \quad \forall x \in B_{R_0}$$

where

$$\begin{aligned} a_0 &:= \frac{\Delta^k v_l(R_0)}{\Delta^k g(R_0)} \\ a_i &:= \frac{\Delta^{k-i} v_l(R_0) - \sum_{j=0}^{i-1} a_j \Delta^{k-i} g_j(R_0)}{\Delta^{k-i} g_i(R_0)}, \quad \forall i \in \{1, 2, \dots, k-1\}, \\ a_k &:= v_l(R_0) - \sum_{i=0}^{k-1} a_i g_i(R_0). \end{aligned}$$

We can check that

$$\begin{aligned} z_l &\in W_{N,rad}^{m,2}(B_{R_0}), \\ \nabla^m v_l &= \nabla^m z_l \text{ in } B_{R_0}. \end{aligned}$$

Combining the proofs when  $m$  is even in [28] and when  $m = 3$ , the Radial Lemma 2.4 and 2.5, we have for  $R_0 \geq 1$

**Lemma 5.1.** *For  $0 < |x| \leq R_0$ , there exists some positive constant  $d(m, R_0)$  depending only on  $m$  and  $R_0$  such that*

$$\begin{aligned} |v_l(|x|)|^2 &\leq |z_l(|x|)|^2 \left( 1 + c_m \frac{1}{R_0} \|\Delta^k v_l\|_2^2 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0} \|v_l\|_{W^{1,2}}^2 \right)^2 \\ &\quad + d(m, R_0). \end{aligned}$$

Now, setting

$$w_l(|x|) := z_l(|x|) \left( 1 + c_m \frac{1}{R_0} \|\Delta^k v_l\|_2^2 + c_m \sum_{j=0}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0} \|v_l\|_{W^{1,2}}^2 \right).$$

Since

$$\begin{aligned} z_l &\in W_{N,rad}^{m,2}(B_{R_0}), \\ \nabla^m v_l &= \nabla^m z_l \text{ in } B_{R_0}. \end{aligned}$$

we have

$$w_l \in W_{N,rad}^{m,2}(B_{R_0})$$

and

$$\|\nabla^m w_l\|_2 = \|\nabla^m z_l\|_2 \left( 1 + c_m \frac{1}{R_0} \|\Delta^k v_l\|_2^2 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0} \|v_l\|_{W^{1,2}}^2 \right).$$

Note that

$$\begin{aligned} \|\nabla^m z_l\|_2 &= \|\nabla^m v_l\|_2 \\ &\leq \left( 1 - \lambda \|\Delta^k v_l\|_2^2 - \lambda \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 - \lambda \|v_l\|_{W^{1,2}}^2 \right)^{1/2} \\ &\leq 1 - \frac{\lambda}{2} \|\Delta^k v_l\|_2^2 - \frac{\lambda}{2} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 - \frac{\lambda}{2} \|v_l\|_{W^{1,2}}^2 \end{aligned}$$

where

$$\lambda = \min \left\{ \binom{m}{j} \tau^{m-j} : j = 0, 1, \dots, m. \right\}$$

we have

$$\begin{aligned} \|\nabla^m w_l\|_2 &\leq \left( 1 - \frac{\lambda}{2} \|\Delta^k v_l\|_2^2 - \frac{\lambda}{2} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 - \frac{\lambda}{2} \|v_l\|_{W^{1,2}}^2 \right) \times \\ &\quad \times \left( 1 + c_m \frac{1}{R_0} \|\Delta^k v_l\|_2^2 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0} \|v_l\|_{W^{1,2}}^2 \right) \\ &\leq 1 \end{aligned}$$

if we choose  $R_0 = R_0(\tau)$  sufficiently large.

Finally, note that

$$I_1 \leq e^{\beta_0 d(m, R_0)} \int_{B_{R_0}} e^{\beta_0 w_l^2} dx,$$

by using Theorem B, we can conclude that  $I_1$  is bounded by a constant depending only on  $\tau$  since  $\|\nabla^m w_l\|_2 \leq 1$ .

Combining the above estimates and applying Fatou's lemma, we can conclude that

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \|\nabla(-\Delta + \tau I)^k u\|_2^2 + \tau \|(-\Delta + \tau I)^k u\|_2^2 \leq 1} \int_{\mathbb{R}^{2m}} (e^{\beta_0 u^2} - 1) dx < \infty.$$

□

**Proofs of Theorem 1.3 and Theorem 1.4 when  $m$  is odd:** From Lemma 2.2, we have the conclusion of Theorem 1.3 when  $m = 2k + 1$ ,  $k \in \mathbb{N}$ .

Again, when  $\beta > \beta_0$ , we can check that the sequence of test functions which gives the sharpness of Adams' inequality in bounded domains in [2] gives also the sharpness of Adams' inequalities in unbounded domains. See Proposition 6.2 in [28].

Moreover, we can choose  $a_j = 1$ ,  $j = 0, \dots, m$  to get the Theorem 1.4.

## 6. PROOF OF THEOREM 1.1

*Proof.* Let  $u \in W^{m, \frac{n}{m}}(\mathbb{R}^n)$  be such that

$$\left\| \nabla (-\Delta + I)^k u \right\|_{\frac{n}{m}}^{\frac{n}{m}} + \left\| (-\Delta + I)^k u \right\|_{\frac{n}{m}}^{\frac{n}{m}} \leq 1,$$

by density arguments, we can find a sequence of functions  $u_l \in C_0^\infty(\mathbb{R}^n)$  such that  $u_l \rightarrow u$  in  $W^{m, \frac{n}{m}}(\mathbb{R}^n)$ ,  $\int_{\mathbb{R}^n} \left( \left| \nabla (-\Delta + I)^k u_l \right|^{\frac{n}{m}} + \left| (-\Delta + I)^k u_l \right|^{\frac{n}{m}} \right) dx \leq 1$  and  $\text{supp } u_l \subset B_{R_l}$  for any fixed  $l$ . Let  $f_l := (-\Delta + I)^k u_l$  and consider the problem

$$\begin{cases} (-\Delta + I)^k v_l = f_l^\# \\ v_l \in W_N^{m, 2}(B_{R_l}) \end{cases}.$$

By the properties of rearrangement, we have

$$\begin{aligned} \int_{B_{R_l}} \left| (-\Delta + I)^k v_l \right|^{\frac{n}{m}} dx &= \int_{B_{R_l}} \left| (-\Delta + I)^k u_l \right|^{\frac{n}{m}} dx \\ \int_{B_{R_l}} \left| \nabla (-\Delta + I)^k v_l \right|^{\frac{n}{m}} dx &\leq \int_{B_{R_l}} \left| \nabla (-\Delta + I)^k u_l \right|^{\frac{n}{m}} dx \end{aligned}$$

Therefore, we have

$$(6.1) \quad \int_{\mathbb{R}^n} \left( \left| \nabla (-\Delta + I)^k v_l \right|^{\frac{n}{m}} + \left| (-\Delta + I)^k v_l \right|^{\frac{n}{m}} \right) dx \leq 1$$

By Corollary 2.1, we get

$$\int_{B_{R_l}} \phi \left( \beta_0 |u_l|^{\frac{n}{n-m}} \right) dx = \int_{B_{R_l}} \phi \left( \beta_0 |u_l^\#|^{\frac{n}{n-m}} \right) dx \leq \int_{B_{R_l}} \phi \left( \beta_0 |v_l|^{\frac{n}{n-m}} \right) dx$$

Here,  $\beta_0 = \beta(n, m)$ .

Again, write

$$\begin{aligned} \int_{B_{R_l}} \phi \left( \beta_0 |v_l|^{\frac{n}{n-m}} \right) dx &\leq \int_{B_{R_0}} \phi \left( \beta_0 |v_l|^{\frac{n}{n-m}} \right) dx + \int_{B_{R_l} \setminus B_{R_0}} \phi \left( \beta_0 |v_l|^{\frac{n}{n-m}} \right) dx \\ &= I_1 + I_2 \end{aligned}$$

where  $R_0$  is a positive constant and will be chosen later. We will prove that both  $I_1$  and  $I_2$  are bounded uniformly.

To do that, again, first, we need to construct an auxiliary radial function  $w_l \in W_N^{m, \frac{n}{m}}(B_{R_0})$  with  $\|\nabla^m w_l\|_{\frac{n}{m}} \leq 1$  which increases the integral  $I_1$ . For each  $i \in \{0, 1, 2, \dots, k-1\}$  we define

$$g_i(|x|) := |x|^{m-1-2i}, \quad \forall x \in B_{R_0}$$

so  $g_i \in W_{rad}^{m, \frac{n}{m}}(B_{R_0})$ . Moreover,

$$\Delta^j g_i(|x|) = \begin{cases} c_i^j |x|^{m-1-2(i+j)} & \text{for } j \in \{1, 2, \dots, k-i\} \\ 0 & \text{for } j \in \{k-i+1, \dots, k\} \end{cases} \quad \forall x \in B_{R_0}$$

where

$$c_i^j = \prod_{h=1}^j [6k - 2(i+h-1)] [2k - 2(i+h-1)], \quad \forall j \in \{1, 2, \dots, k-i\}.$$

Let

$$z_l(|x|) := v_l(|x|) - \sum_{i=0}^{k-1} a_i g_i(|x|) - a_k, \quad \forall x \in B_{R_0}$$

where

$$\begin{aligned} a_0 &:= \frac{\Delta^k v_l(R_0)}{\Delta^k g(R_0)} \\ a_i &:= \frac{\Delta^{k-i} v_l(R_0) - \sum_{j=0}^{i-1} a_j \Delta^{k-i} g_j(R_0)}{\Delta^{k-i} g_i(R_0)}, \quad \forall i \in \{1, 2, \dots, k-1\}, \\ a_k &:= v_l(R_0) - \sum_{i=0}^{k-1} a_i g_i(R_0). \end{aligned}$$

We can check that

$$\begin{aligned} z_l &\in W_{N,rad}^{m,\frac{n}{m}}(B_{R_0}), \\ \nabla^m v_l &= \nabla^m z_l \text{ in } B_{R_0}. \end{aligned}$$

By a similar argument to that in [28], and a combination of Radial Lemmas 2.4 and 2.5, we can prove that for  $R_0 \geq 1$

**Lemma 6.1.** *For  $0 < |x| \leq R_0$  we have for some constant  $d(m, n, R_0)$  such that*

$$\begin{aligned} &|v_l(|x|)|^{\frac{n}{m}} \\ &\leq |z_l(|x|)|^{\frac{n}{m}} \left( 1 + c_{m,n} \frac{1}{R_0} \|\Delta^k v_l\|_{\frac{n}{m}}^{\frac{n}{m}} + c_{m,n} \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,n}}{R_0} \|v_l\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}} \right)^{\frac{n}{m}} \\ &\quad + d(m, n, R_0). \end{aligned}$$

Now, setting

$$w_l(|x|) := z_l(|x|) \left( 1 + c_{m,n} \frac{1}{R_0} \|\Delta^k v_l\|_{\frac{n}{m}}^{\frac{n}{m}} + c_{m,n} \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,n}}{R_0} \|v_l\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}} \right).$$

Since

$$\begin{aligned} z_l &\in W_{N,rad}^{m,\frac{n}{m}}(B_{R_0}), \\ \nabla^m v_l &= \nabla^m z_l \text{ in } B_{R_0}. \end{aligned}$$

we have

$$w_l \in W_{N,rad}^{m,\frac{n}{m}}(B_{R_0})$$

and

$$\|\nabla^m w_l\|_{\frac{n}{m}} = \|\nabla^m z_l\|_{\frac{n}{m}} \left( 1 + \frac{c_{m,n}}{R_0} \|\Delta^k v_l\|_{\frac{n}{m}}^{\frac{n}{m}} + \sum_{j=1}^{k-1} \frac{c_{m,n}}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,n}}{R_0} \|v_l\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}} \right).$$

Note that from Lemma 2.3:

$$\begin{aligned}
\|\nabla^m z_l\|_{\frac{n}{m}} &= \|\nabla^m v_l\|_{\frac{n}{m}} \\
&\leq \left(1 - \frac{1}{C} \|\Delta^k v_l\|_{\frac{n}{m}} - \frac{1}{C} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1, \frac{n}{m}}} - \frac{1}{C} \|v_l\|_{W^{1, \frac{n}{m}}}\right)^{m/n} \\
&\leq 1 - \frac{m}{nC} \|\Delta^k v_l\|_{\frac{n}{m}} - \frac{m}{nC} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1, \frac{n}{m}}} - \frac{m}{nC} \|v_l\|_{W^{1, \frac{n}{m}}},
\end{aligned}$$

we have

$$\begin{aligned}
\|\nabla^m w_l\|_{\frac{n}{m}} &\leq \left(1 - \frac{m}{nC} \|\Delta^k v_l\|_{\frac{n}{m}} - \frac{m}{nC} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{W^{1, \frac{n}{m}}} - \frac{m}{nC} \|v_l\|_{W^{1, \frac{n}{m}}}\right) \times \\
&\quad \times \left(1 + c_{m,n} \frac{1}{R_0} \|\Delta^k v_l\|_{\frac{n}{m}} + c_{m,n} \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1, \frac{n}{m}}} + \frac{c_{m,n}}{R_0} \|v_l\|_{W^{1, \frac{n}{m}}}\right) \\
&\leq 1
\end{aligned}$$

if we choose  $R_0$  sufficiently large.

Finally, note that

$$I_1 \leq e^{\beta_0 d(m,n,R_0)} \int_{B_{R_0}} e^{\beta_0 w_l^2} dx,$$

by using Theorem B, we can conclude that  $I_1$  is bounded by a constant depending only on  $n$  and  $m$ .

Now, by the same argument as in [28] and noting that from (6.1) and Lemma 2.3, we have  $\|v_l\|_{W^{1, \frac{n}{m}}} \leq D$  for some constant  $D > 0$ , we can conclude that  $I_2$  is also bounded by a constant depending only on  $n$  and  $m$ .

Combining the above estimates and employing Fatou's lemma, we can conclude that

$$\sup_{u \in W^{m, \frac{n}{m}}(\mathbb{R}^n), \|\nabla(-\Delta+I)^k u\|_{\frac{n}{m}} + \|(-\Delta+I)^k u\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \phi\left(\beta(n, m) |u|^{\frac{n}{n-m}}\right) dx < \infty.$$

Again, when  $\beta > \beta(n, m)$ , it is showed by Kozono, Sato and Wadade [19] that the supremum is infinite. In fact, the sequence of test functions which gives the sharpness of Adams' inequality in bounded domains in [2] gives also the sharpness of Adams' inequality in unbounded domains (see Proposition 6.2 in [28]).  $\square$

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